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# Representation and approximation of multivariate functions with mixed smoothness by hyperbolic wavelets

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## Abstract

In this paper, we study the representation theorems of multivariate functions with mixed smoothness by wavelet basis formed by tensor products of univariate wavelets, we also study the best approximation in the  $L_q(\mathbb{R}^d)$  metric for some function classes with mixed smoothness by hyperbolic wavelets and obtain some asymptotic estimates of approximating order.

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**Keywords:** Hyperbolic wavelets; Functions with mixed smoothness; Decomposition of functions; Best approximation

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## 1. Introduction

Let  $\varphi$  be a univariate scaling function that satisfies multiresolution analysis of  $L_2(\mathbb{R})$ , i.e., a nested sequence of closed subspaces  $\{V_m\}_{m \in \mathbb{Z}^d}$  of  $L_2(\mathbb{R})$  such that

- (i)  $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L_2(\mathbb{R})$ ,
- (ii)  $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ , and  $\bigcup_{j \in \mathbb{Z}} V_j$  is dense in  $L_2(\mathbb{R})$ ,
- (iii) For any  $j \in \mathbb{Z}$ ,  $f(\cdot) \in V_j$  if and only if  $f(2 \cdot) \in V_{j+1}$ ,
- (iv)  $\{\varphi(\cdot - j)\}_{j \in \mathbb{Z}}$  is an orthonormal basis of  $V_0$ .

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We define the subspace  $W_j$  as an orthogonal complement of  $V_j$  in  $V_{j+1}$ , and the function  $\psi$ , the wavelet, such that the set of functions  $\{\psi(\cdot - j)\}_{j \in \mathbb{Z}}$  is an orthonormal basis of  $W_0$ . Then we can represent the space  $L_2(\mathbb{R})$  as a direct sum

$$L_2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j = V_1 \bigoplus_{j \geq 1} W_j.$$

So we can construct efficient basis for  $L_2(\mathbb{R})$  and other function spaces by dilation and shifts. For example, the functions

$$\psi_{j,k}(\cdot) := 2^{k/2} \psi(2^k \cdot - j), \quad j, k \in \mathbb{Z} \quad (1.1)$$

and

$$\psi_{j,s}^* := \begin{cases} \psi_{j,s} := 2^{s/2} \psi(2^s \cdot - j), & \text{if } s > 0, \\ \varphi_{j,1} := 2^{1/2} \varphi(2^1 \cdot - j), & \text{if } s = 0, \end{cases} \quad j, s \in \mathbb{Z}, s > 0, \quad (1.2)$$

both form a orthonormal basis for  $L_2(\mathbb{R})$  (see [3,9]). We also suppose  $\varphi, \psi$  are  $l$ -regular, i.e.,  $\varphi, \psi$  satisfy

$$|\varphi^{(k)}(x)|, |\psi^{(k)}(x)| \leq C_{p,k} (1 + |x|)^{-p}, \quad k = 0, 1, \dots, l, \quad p \in \mathbb{Z}_+, \quad x \in \mathbb{R}. \quad (1.3)$$

There exists a different indexing for the functions  $\psi_{j,k}$ . Let  $\mathcal{D}(\mathbb{R})$  denote the set of dyadic intervals. Each such interval  $I$  is the form  $I = [j2^{-k}, (j+1)2^{-k}]$ . We also denote by  $\mathcal{D}_+(\mathbb{R})$  the subset of  $\mathcal{D}(\mathbb{R})$  such that for each interval  $I \in \mathcal{D}_+(\mathbb{R})$ , the length  $|I|$  of  $I$  satisfies  $|I| \leq 1$ . For each dyadic interval  $I \in \mathcal{D}(\mathbb{R})$ , we define

$$\psi_I := \psi_{j,k}, \quad I = [j2^{-k}, (j+1)2^{-k}] \in \mathcal{D}(\mathbb{R}) \quad (1.4)$$

and

$$\psi_I^* := \psi_{j,s}^*, \quad I = [j2^{-s}, (j+1)2^{-s}] \in \mathcal{D}_+(\mathbb{R}), \quad (1.5)$$

thus the basis  $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$  and the basis  $\{\psi_{j,s}^*\}_{j,s \in \mathbb{Z}, s \geq 0}$  are the same as  $\{\psi_I\}_{I \in \mathcal{D}(\mathbb{R})}$  and  $\{\psi_I^*\}_{I \in \mathcal{D}_+(\mathbb{R})}$ .

For multivariate function space  $L_2(\mathbb{R}^d)$ , we can construct multivariate wavelet basis by taking tensor products of the univariate basis functions. If  $\psi$  is a univariate wavelet and  $d \geq 1$ , then for  $\mathbf{j}, \mathbf{k}, \mathbf{s} \in \mathbb{Z}^d$ ,  $\mathbf{j} = (j_1, \dots, j_d)$ ,  $\mathbf{k} = (k_1, \dots, k_d)$ ,  $\mathbf{s} = (s_1, \dots, s_d) \geq 0$  (i.e.,  $s_i \geq 0$ ,  $i = 1, \dots, d$ ), and  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , the functions

$$\psi_{\mathbf{j},\mathbf{k}}(x) := \psi_{j_1,k_1}(x_1) \cdots \psi_{j_d,k_d}(x_d) \quad (1.6)$$

and

$$\psi_{\mathbf{j},\mathbf{s}}^*(x) := \psi_{j_1,s_1}^*(x_1) \cdots \psi_{j_d,s_d}^*(x_d) \quad (1.7)$$

are both the orthonormal basis for  $L_2(\mathbb{R}^d)$ .

Again, we can use another indexing for the basis functions (1.6) and (1.7). Denote by  $\mathcal{D}(\mathbb{R}^d)$  the set of all dyadic rectangles in  $\mathbb{R}^d$ , any  $\mathbf{I} \in \mathcal{D}(\mathbb{R}^d)$  is the form  $\mathbf{I} = I_1 \times \cdots \times I_d$  with  $I_1, \dots, I_d \in \mathcal{D}(\mathbb{R})$ . Also let

$$\mathcal{D}_+(\mathbb{R}^d) := \{\mathbf{I} \in \mathcal{D}(\mathbb{R}^d) : \mathbf{I} = I_1 \times \cdots \times I_d, \quad I_1, \dots, I_d \in \mathcal{D}_+(\mathbb{R})\}.$$

We define

$$\psi_{\mathbf{I}}(x) := \psi_{I_1}(x_1) \cdots \psi_{I_d}(x_d), \quad \mathbf{I} \in \mathcal{D}(\mathbb{R}^d) \quad (1.8)$$

and

$$\psi_{\mathbf{I}}^*(x) := \psi_{I_1}^*(x_1) \cdots \psi_{I_d}^*(x_d), \quad \mathbf{I} \in \mathcal{D}_+(\mathbb{R}^d). \quad (1.9)$$

Therefore, the wavelet basis (1.6) and (1.7) are the same as the set of functions  $\{\psi_{\mathbf{I}}\}_{\mathbf{I} \in \mathcal{D}(\mathbb{R}^d)}$  and  $\{\psi_{\mathbf{I}}^*\}_{\mathbf{I} \in \mathcal{D}_+(\mathbb{R}^d)}$ .

For  $\mathbf{k}, \mathbf{s} \in \mathbb{Z}^d$ ,  $\mathbf{s} \geq 0$ , and  $1 \leq p \leq \infty$ , let

$$W_{\mathbf{k}}(p) := \overline{\text{span}\{\psi_{\mathbf{j},\mathbf{k}} : \mathbf{j} \in \mathbb{Z}^d\}}, \quad W_{\mathbf{s}}^*(p) := \overline{\text{span}\{\psi_{\mathbf{j},\mathbf{s}}^* : \mathbf{j} \in \mathbb{Z}^d\}} \quad (1.10)$$

denote respectively the closed linear span of the finite linear combinations of the functions  $\psi_{\mathbf{j},\mathbf{k}}$ ,  $\mathbf{j} \in \mathbb{Z}^d$  and  $\psi_{\mathbf{j},\mathbf{s}}^*$ ,  $\mathbf{j} \in \mathbb{Z}^d$ , with the closure taken with respect to the  $L_p(\mathbb{R}^d)$ -norm. We also suppose that  $D_{\mathbf{k}}$  and  $D_{\mathbf{s}}^*$  are the corresponding (orthogonal) projection operators, that is, for any  $f \in L_p(\mathbb{R}^d)$ , we have

$$D_{\mathbf{k}}f(x) := \sum_{\mathbf{j} \in \mathbb{Z}^d} d_{\mathbf{j},\mathbf{k}}(f) \psi_{\mathbf{j},\mathbf{k}}(x), \quad d_{\mathbf{j},\mathbf{k}}(f) := \int_{\mathbb{R}^d} f(x) \psi_{\mathbf{j},\mathbf{k}}(x) dx \quad (1.11)$$

and

$$D_{\mathbf{s}}^*f(x) := \sum_{\mathbf{j} \in \mathbb{Z}^d} d_{\mathbf{j},\mathbf{s}}^*(f) \psi_{\mathbf{j},\mathbf{s}}^*(x), \quad d_{\mathbf{j},\mathbf{s}}^*(f) := \int_{\mathbb{R}^d} f(x) \psi_{\mathbf{j},\mathbf{s}}^*(x) dx. \quad (1.12)$$

For  $1 \leq p \leq \infty$ ,  $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{R}^d$ ,  $\gamma > 0$ ,  $n = 0, 1, \dots$ , let

$$\mathcal{H}_n^\gamma := \mathcal{H}_n^\gamma(L_p(\mathbb{R}^d)) := \overline{\text{span}\{\psi_{\mathbf{j},\mathbf{k}} : \mathbf{j}, \mathbf{k} \in \mathbb{Z}^d, (\mathbf{k}, \gamma) \leq n\}} \quad (1.13)$$

and

$$\mathcal{H}_n^{\gamma,*} := \mathcal{H}_n^{\gamma,*}(L_p(\mathbb{R}^d)) := \overline{\text{span}\{\psi_{\mathbf{j},\mathbf{s}}^* : \mathbf{j}, \mathbf{s} \in \mathbb{Z}^d, \mathbf{s} \geq 0, (\mathbf{s}, \gamma) \leq n\}} \quad (1.14)$$

denote respectively the  $L_p(\mathbb{R}^d)$ -closure of linear span of the functions  $\psi_{\mathbf{j},\mathbf{k}}$ ,  $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^d$ ,  $(\mathbf{k}, \gamma) \leq n$  and  $\psi_{\mathbf{j},\mathbf{s}}^*$ ,  $\mathbf{j}, \mathbf{s} \in \mathbb{Z}^d$ ,  $\mathbf{s} \geq 0$ ,  $(\mathbf{s}, \gamma) \leq n$ , where  $(\mathbf{k}, \gamma) := k_1\gamma_1 + \dots + k_d\gamma_d$ .

We are interested in the approximation properties of the spaces  $\mathcal{H}_n^\gamma$  and  $\mathcal{H}_n^{\gamma,*}$ , we call the approximation by  $\mathcal{H}_n^\gamma$  or  $\mathcal{H}_n^{\gamma,*}$  hyperbolic wavelet approximation in analogy with the approximation by trigonometric polynomial approximation (see [14]). For  $1 \leq p \leq \infty$ ,  $f \in L_p(\mathbb{R}^d)$ , we define

$$\begin{aligned} E_n^\gamma(f)_p &:= E(f, \mathcal{H}_n^\gamma)_p := \inf_{g \in \mathcal{H}_n^\gamma} \|f - g\|_p, \\ E_n^{\gamma,*}(f)_p &:= E(f, \mathcal{H}_n^{\gamma,*})_p := \inf_{g \in \mathcal{H}_n^{\gamma,*}} \|f - g\|_p \end{aligned} \quad (1.15)$$

with  $\|\cdot\|_p$  here and later the  $L_p(\mathbb{R}^d)$  norm. From the fact  $\mathcal{H}_n^{\gamma,*} \subset \mathcal{H}_n^\gamma$ , we know that for any  $f \in L_p(\mathbb{R}^d)$ , we have

$$E_n^\gamma(f)_p \leq E_n^{\gamma,*}(f)_p. \quad (1.16)$$

We also define the partial sum operators of hyperbolic wavelets as follows:

$$S_n^\gamma f(x) := \sum_{(\mathbf{k}, \gamma) \leq n} D_{\mathbf{k}} f(x), \quad S_n^{\gamma,*} f(x) := \sum_{\mathbf{s} \geq 0, (\mathbf{s}, \gamma) \leq n} D_{\mathbf{s}}^* f(x). \quad (1.17)$$

For  $1 < p < \infty$ , by Littlewood–Paley theory about hyperbolic wavelets (see [4]), we can show that the operators  $S_n^\gamma, S_n^{\gamma,*}$  are bounded from  $L_p(\mathbb{R}^d)$  to  $\mathcal{H}_n^\gamma, \mathcal{H}_n^{\gamma,*}$ , and also

$$E_n^\gamma(f)_p \asymp \|f - S_n^\gamma f\|_p, \quad E_n^{\gamma,*}(f)_p \asymp \|f - S_n^{\gamma,*} f\|_p, \quad (1.18)$$

where  $A \asymp B$  means  $A \ll B$  and  $B \ll A$ , and  $A \ll B$  means there exists a positive constant  $c$  such that  $A \leq cB$ .

We are interested in the approximating properties of the function classes with mixed smoothness by hyperbolic wavelets. The investigation of function classes with mixed smoothness defined on  $\mathbb{R}^d$  as well as on  $T^d$  was initiated by S.M. Nikolskii [12, p. 390]. In 1963–1965, P.I. Lizorkin and S.M. Nikolskii [7], S.M. Nikolskii [11], T.I. Amanov [1] defined three types of function spaces with mixed smoothness, i.e., the Sobolev-type space  $S_p^r L$ , the Hölder-type space  $S_p^r H$ , and the Besov-type space  $S_{p,\theta}^r B$ . They obtained a series of fundamental results about these spaces, such as the representation theorem, imbedding theorem, trace theorem, etc. In [8] P.I. Lizorkin and S.M. Nikolskii studied these spaces again from the view point of function decomposition. They obtained the representation theorem by Littlewood–Paley blocks, introduced the class of entire functions whose spectrals lie in a step hyperbolic cross, and proposed to study the approximation by hyperbolic cross. It is well known that in the periodic case approximation by trigonometric polynomials whose spectrals lie in some hyperbolic crosses was first considered by K.I. Babenko [2]. In the decades of the 1970s and 1980s, N.S. Nikolskaja [10], E.M. Galeev [6], D. Zung [5], V.N. Temlyakov [14], etc., systematically investigated the multivariate approximation by trigonometric polynomials of hyperbolic crosses in the periodic case. As to the  $\mathbb{R}^d$  (non-periodic) case, Wang Heping and Sun Yongsheng [15] studied the approximation of multivariate functions with mixed smoothness by entire functions whose spectrals lie in a step hyperbolic cross; and there are very few works about the  $d \geq 2$  case as far as we know. In this paper, we will study the best approximation in  $L_q(\mathbb{R}^d)$  metric for the function classes  $S_p^r L(\mathbb{R}^d), S_p^r H(\mathbb{R}^d), S_{p,\theta}^r B(\mathbb{R}^d)$ ,  $1 < p \leq q < \infty$  by hyperbolic wavelets. Let us recall the definitions of the spaces with mixed smoothness (see [8] or [15]).

For  $\forall e \subset e_d := \{1, 2, \dots, d\}$ ,  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , we define

$$x^e := (x_1^e, \dots, x_d^e), \quad x_i^e := \begin{cases} x_i, & i \in e, \\ 0, & i \notin e. \end{cases}$$

For  $r = (r_1, \dots, r_d) \geq 0$ , let

$$D^r f(x) := \frac{\partial^{|r|} f(x)}{\partial x_1^{r_1} \dots \partial x_d^{r_d}} \quad (|r| := r_1 + \dots + r_d)$$

be the generalized derivative of  $f$  in the sense of Liouville (see [7]). Then for  $r = (r_1, \dots, r_d) > 0$ , the Sobolev space  $S_p^r L(\mathbb{R}^d)$  with mixed smoothness is defined as follows:

$$S_p^r L(\mathbb{R}^d) := \left\{ f \in L_p(\mathbb{R}^d) : \|f\|_{S_p^r L} := \sum_{e \subset e_d} \|D^{r^e} f\|_p < \infty \right\}. \quad (1.19)$$

For  $f \in L_q(\mathbb{R}^d)$ ,  $t = (t_1, \dots, t_d) > 0$ ,  $e \subset e_d$ , we set

$$\Omega^{l^e}(f, t^e)_q := \sup_{|h_j| \leq t_j} \|\Delta_{h^e}^{l^e} f(\cdot)\|_q, \quad (1.20)$$

where  $l \in \mathbb{Z}_+$  is a fixed positive integer,  $h = (h_1, \dots, h_d) \in \mathbb{R}^d$ , and

$$\begin{aligned} \Delta_{h_j, j}^0 f(x) &:= f(x), \\ \Delta_{h_j, j}^l f(x) &:= \sum_{k=0}^l (-1)^{l-k} \binom{l}{k} f(x_1, \dots, x_j + kh_j, \dots, x_d), \\ \Delta_{h^e}^{l^e} f(x) &:= \left( \prod_{j \in e} \Delta_{h_j, j}^l \right) f(x), \quad \Delta_h^l f(x) := \Delta_{h_1, 1}^l \cdots \Delta_{h_d, d}^l f(x). \end{aligned} \quad (1.21)$$

As we know,  $\Delta_{h^e}^{l^e} f(x)$  is the  $l$ -order mixed difference with vector-valued step  $h$  of  $f(x)$ ,  $\Omega^{l^e}(f, t^e)_q$  is the  $l$ -order mixed moduli of smoothness in  $L_q$  norm. For  $r = (r_1, \dots, r_d) > 0$ , we choose  $l \in \mathbb{Z}_+$  such that  $l > \max\{r_1, \dots, r_d\}$ , then the Hölder space  $S_p^r H(\mathbb{R}^d)$  and the Besov space  $S_{p, \theta}^r B(\mathbb{R}^d)$  with mixed smoothness are defined in the following way respectively [8, 15]:

$$\begin{aligned} S_p^r H(\mathbb{R}^d) &:= S_{p, \infty}^r B(\mathbb{R}^d) \\ &:= \left\{ f \in L_p(\mathbb{R}^d) : \|f\|_{S_p^r H} := \|f\|_{S_{p, \infty}^r B} \right. \\ &\quad \left. := \sum_{e \subset e_d} \sup_{t > 0} \prod_{j \in e} t_j^{-r_j} \Omega^{l^e}(f, t^e)_p < \infty \right\}, \end{aligned} \quad (1.22)$$

$$\begin{aligned} S_{p, \theta}^r B(\mathbb{R}^d) &:= \left\{ f \in L_p(\mathbb{R}^d) : \|f\|_{S_{p, \theta}^r B} := \sum_{e \subset e_d} \left( \int_0^2 \cdots \int_0^2 \prod_{j \in e} t_j^{-\theta r_j - 1} \right. \right. \\ &\quad \left. \left. \cdot (\Omega^{l^e}(f, t^e)_p)^\theta \prod_{j \in e} dt_j \right)^{1/\theta} < \infty \right\}, \quad (1 \leq \theta < \infty). \end{aligned} \quad (1.23)$$

It is to be noticed that when  $e$  is empty set, the corresponding terms in (1.20), (1.22), (1.23) are  $\|f\|_p$  by definition, and  $\|f\|_{S_p^r L}$ ,  $\|f\|_{S_p^r H}$ ,  $\|f\|_{S_{p, \theta}^r B}$  are norms. From the definitions we know that the Hölder space is the special case of the Besov space when  $\theta = \infty$ , so it is enough to consider Sobolev spaces and Besov spaces only. Denote by  $S_p^r L$ ,  $S_{p, \theta}^r B$  the unit balls of the spaces  $S_p^r L(\mathbb{R}^d)$  and  $S_{p, \theta}^r B(\mathbb{R}^d)$ . For  $1 \leq p, q \leq \infty$ ,  $1 \leq \theta \leq \infty$ ,  $r = r_1 \cdot \gamma > 0$ ,  $r_1 > 0$ ,  $\gamma \in \mathbb{R}^d$ ,  $n \geq 1$ , we introduce the quantities

$$E_n^\gamma(S_p^r L)_q = \sup_{\|f\|_{S_p^r L} \leq 1} E_n^\gamma(f)_q, \quad E_n^{\gamma, *}(S_p^r L)_q = \sup_{\|f\|_{S_p^r L} \leq 1} E_n^{\gamma, *}(f)_q, \quad (1.24)$$

$$E_n^\gamma(S_{p, \theta}^r B)_q = \sup_{\|f\|_{S_{p, \theta}^r B} \leq 1} E_n^\gamma(f)_q, \quad E_n^{\gamma, *}(S_{p, \theta}^r B)_q = \sup_{\|f\|_{S_{p, \theta}^r B} \leq 1} E_n^{\gamma, *}(f)_q. \quad (1.25)$$

In Section 4, we will find out the asymptotically exact order of  $E_n^\gamma(S_p^r L)_q$ ,  $E_n^{\gamma,*}(S_p^r L)_q$ , and  $E_n^\gamma(S_{p,\theta}^r B)_q$ ,  $E_n^{\gamma,*}(S_{p,\theta}^r B)_q$  when  $1 < p \leq q < \infty$ ,  $1 \leq \theta \leq \infty$ ,  $n \rightarrow \infty$ . These results may be compared with their periodic analogy obtained by V.N. Temlyakov [14] and Romaniuk [13]. The main tool we use is the representation theory of functions with mixed smoothness by hyperbolic wavelets, which we will show in Section 3. In Section 2, we shall give some properties of operators  $D_s^*$  and  $D_k$  which will be used in Section 3. In this paper, we restrict our development to approximation on  $\mathbb{R}^d$ , we could also give similar results for the case of approximation on a compact subset of  $\mathbb{R}^d$  or on the torus  $T^d$ .

## 2. Properties of operators $D_k$ and $D_s^*$ ( $s \geq 0$ )

In the following, we always suppose that  $\varphi, \psi$  are  $l$ -regular,  $\mathbf{k}, \mathbf{s} \in \mathbb{Z}^d$ ,  $\mathbf{s} \geq 0$ ,  $r = (r_1, \dots, r_d) \in \mathbb{R}^d$ ,  $0 \leq r_1, \dots, r_d \leq l$ .

**Lemma 2.1.** *Let  $\mathbf{k}, r \in \mathbb{Z}^d$  fixed,  $1 \leq p \leq \infty$ . Then for any finite sum  $f(x) = \sum_{\mathbf{j}} d_{\mathbf{j},\mathbf{k}} \times \psi_{\mathbf{j},\mathbf{k}}(x)$ , we have*

$$\|D^r f\|_p \asymp 2^{(1/2-1/p)|\mathbf{k}|} 2^{(r,\mathbf{k})} \left( \sum_{\mathbf{j}} |d_{\mathbf{j},\mathbf{k}}|^p \right)^{1/p}. \quad (2.1)$$

**Proof.** From

$$D^r(\psi_{\mathbf{j},\mathbf{k}})(x) = 2^{(r,\mathbf{k})} (D^r \psi)_{\mathbf{j},\mathbf{k}}(x) := 2^{(r,\mathbf{k})} \prod_{j=i}^d \psi_{j_i, k_i}^{(r_i)}(x_i),$$

we know that

$$D^r f(x) = \sum_{\mathbf{j}} d_{\mathbf{j},\mathbf{k}} D^r(\psi_{\mathbf{j},\mathbf{k}})(x) = \sum_{\mathbf{j}} d_{\mathbf{j},\mathbf{k}} 2^{(r,\mathbf{k})} (D^r \psi)_{\mathbf{j},\mathbf{k}}(x).$$

By a simple variable transformation, we know that it suffices to prove the lemma in the case  $\mathbf{k} = \mathbf{0} = (0, \dots, 0)$ .

*Upper estimates of  $\|D^r f\|_p$ .* As  $\psi$  is  $l$ -regular, we have

$$\begin{aligned} C_r(\psi) &:= \sup_{x \in \mathbb{R}^d} \sum_{\mathbf{j} \in \mathbb{Z}^d} |(D^r \psi)_{\mathbf{j},\mathbf{0}}(x)| \leq C \sup_{x \in \mathbb{R}^d} \sum_{\mathbf{j} \in \mathbb{Z}^d} \prod_{i=1}^d (1 + |x_i - j_i|)^{-2} \\ &\leq 2C \sup_{x \in \mathbb{R}^d} \prod_{i=1}^d \sum_{j_i \in \mathbb{Z}} (1 + |j_i|)^{-2} < \infty. \end{aligned}$$

For  $1 \leq p \leq \infty$ , let  $p'$  satisfy  $1/p + 1/p' = 1$ . Then

$$|D^r f(x)| \leq \sum_{\mathbf{j}} |d_{\mathbf{j},\mathbf{0}}| \cdot |(D^r \psi)_{\mathbf{j},\mathbf{0}}(x)|^{1/p} \cdot |(D^r \psi)_{\mathbf{j},\mathbf{0}}(x)|^{1/p'}$$

$$\begin{aligned} &\leq \left( \sum_{\mathbf{j}} |d_{\mathbf{j},\mathbf{0}}|^p \cdot |(D^r \psi)_{\mathbf{j},\mathbf{0}}(x)| \right)^{1/p} \cdot \left( \sum_{\mathbf{j}} |(D^r \psi)_{\mathbf{j},\mathbf{0}}(x)| \right)^{1/p'} \\ &\leq C_r(\psi)^{1/p'} \cdot \left( \sum_{\mathbf{j}} |d_{\mathbf{j},\mathbf{0}}|^p \cdot |(D^r \psi)_{\mathbf{j},\mathbf{0}}(x)| \right)^{1/p}, \end{aligned}$$

since

$$C_r(\psi) < \infty, \quad \int_{\mathbb{R}^d} |(D^r \psi)_{\mathbf{j},\mathbf{0}}(x)| dx = \|(D^r \psi)_{\mathbf{0},\mathbf{0}}\|_1 = \prod_{j=1}^d \|\psi^{(r_j)}\|_{L_1(\mathbb{R})} < \infty,$$

we get

$$\|D^r f\|_p \leq C_r(\psi)^{1/p'} \cdot \|(D^r \psi)_{\mathbf{0},\mathbf{0}}\|_1^{1/p} \cdot \left( \sum_{\mathbf{j}} |d_{\mathbf{j},\mathbf{k}}|^p \right)^{1/p}.$$

*Lower estimate of  $\|D^r f\|_p$ .* Since the wavelet function  $\psi$  is  $l$ -regular, then  $\psi$  has up to  $l$  order of vanishing moments (see [9]), so we can integrate the univariate function  $\psi$ ,  $r_j$  times to find a function  $\mu_j \in L_{p'}(\mathbb{R})$  which satisfies  $(-1)^{r_j} \mu_j^{(r_j)} = \psi$ . It follows that  $\mu_j(x) (j = 1, \dots, d)$  are rapidly decreasing and

$$D^r(\mu_{\mathbf{j},\mathbf{k}})(x) := D^r \left( \prod_{i=1}^d (\mu_i)_{j_i, k_i}(x_i) \right) = (-1)^{|r|} 2^{(r,\mathbf{k})} \psi_{\mathbf{j},\mathbf{k}}(x), \quad \mathbf{j}, \mathbf{k} \in \mathbb{Z}^d.$$

Integration by parts then shows that

$$\int_{\mathbb{R}^d} D^r(\psi_{\mathbf{j},\mathbf{k}}(x)) \cdot \mu_{\mathbf{j}',\mathbf{k}'}(x) dx = 2^{(r,\mathbf{k})} \delta(\mathbf{j}, \mathbf{j}') \delta(\mathbf{k}, \mathbf{k}'),$$

where  $\delta$  is the Kronecker delta. Hence,

$$d_{\mathbf{j},\mathbf{k}} = \int_{\mathbb{R}^d} f(x) \psi_{\mathbf{j},\mathbf{k}}(x) dx = 2^{-(r,\mathbf{k})} \int_{\mathbb{R}^d} D^r f(x) \mu_{\mathbf{j},\mathbf{k}}(x) dx.$$

Since  $\mu_j(x) (j = 1, \dots, d)$  are rapidly decreasing, then

$$C_r(\mu) := \sup_{x \in \mathbb{R}^d} \sum_{\mathbf{j} \in \mathbb{Z}^d} |\mu_{\mathbf{j},\mathbf{0}}(x)| \leq C \sup_{x \in \mathbb{R}^d} \sum_{\mathbf{j} \in \mathbb{Z}^d} \prod_{i=1}^d (1 + |x_i - j_i|)^{-2} < \infty.$$

So we have

$$|d_{\mathbf{j},\mathbf{0}}| \leq \int_{\mathbb{R}^d} |D^r f(x) \mu_{\mathbf{j},\mathbf{0}}(x)| dx = \int_{\mathbb{R}^d} |D^r f(x)| |\mu_{\mathbf{j},\mathbf{0}}(x)|^{1/p} \cdot |\mu_{\mathbf{j},\mathbf{0}}(x)|^{1/p'} dx,$$

then by Hölder inequality, we get

$$|d_{\mathbf{j},\mathbf{0}}| \leq \|\mu_{\mathbf{0},\mathbf{0}}\|_1^{1/p'} \left( \int_{\mathbb{R}^d} |D^r f(x)|^p |\mu_{\mathbf{j},\mathbf{0}}(x)| dx \right)^{1/p}.$$

Hence

$$\begin{aligned} \sum_{\mathbf{j}} |d_{\mathbf{j},\mathbf{0}}|^p &\leq \|\mu_{\mathbf{0},\mathbf{0}}\|_1^{p/p'} \cdot \sum_{\mathbf{j}} \int_{\mathbb{R}^d} |D^r f(x)|^p |\mu_{\mathbf{j},\mathbf{0}}(x)| dx \\ &\leq \|\mu_{\mathbf{0},\mathbf{0}}\|_1^{p/p'} \cdot C_r(\mu) \cdot \|D^r f\|_p^p. \end{aligned}$$

Lemma 2.1 is proved.  $\square$

**Remark 2.1.** Lemma 2.1 holds for any  $f \in W_{\mathbf{k}}(p)$ .

**Remark 2.2.** For  $1 \leq p \leq \infty$ ,  $f(x) = \sum_{\mathbf{j} \in \mathbb{Z}^d} d_{\mathbf{j},\mathbf{s}}^* \psi_{\mathbf{j},\mathbf{s}}^*(x) \in W_{\mathbf{s}}^*(p)$ ,  $\mathbf{s} \geq 0$ , we have

$$\|D^r f\|_p \ll 2^{(1/2-1/p)|\mathbf{s}|} 2^{(r,\mathbf{s})} \left( \sum_{\mathbf{j}} |d_{\mathbf{j},\mathbf{s}}^*|^p \right)^{1/p}. \quad (2.2)$$

Furthermore, if when  $i \in ce(\mathbf{s}) := \{j \in e_d: s_j = 0\}$ ,  $r_i = 0$ , then we also have

$$\|D^r f\|_p \asymp 2^{(1/2-1/p)|\mathbf{s}|} 2^{(r,\mathbf{s})} \left( \sum_{\mathbf{j}} |d_{\mathbf{j},\mathbf{s}}^*|^p \right)^{1/p}. \quad (2.3)$$

The proof is similar, we omit it.

From Lemma 2.1 and the following remarks, we get the corresponding Bernstein inequality and Nikolskii inequality in  $W_{\mathbf{k}}(p)$  or  $W_{\mathbf{s}}^*(p)$ .

**Lemma 2.2.** Let  $1 \leq p \leq q \leq \infty$ ,  $f(x) \in W_{\mathbf{k}}(p)$  or  $f(x) \in W_{\mathbf{k}}^*(p)$ ,  $\mathbf{k} \geq 0$ . Then

$$\|D^r f\|_p \ll 2^{(r,\mathbf{k})} \|f\|_p, \quad \|f\|_q \ll 2^{(1/p-1/q)|\mathbf{k}|} \|f\|_p. \quad (2.4)$$

**Lemma 2.3.** Let  $1 \leq p \leq \infty$ , and when  $i \in ce(\mathbf{s})$ ,  $r_i = 0$ . Then for any  $f$ ,  $D^r f \in L_p(\mathbb{R}^d)$ , we have

$$\|D_{\mathbf{s}}^* f\|_p \ll 2^{-(r,\mathbf{s})} \|D^r f\|_p. \quad (2.5)$$

**Proof.** By Remark 2.2, we get

$$\|D_{\mathbf{s}}^* f\|_p^p \asymp 2^{p(1/2-1/p)|\mathbf{s}|} \sum_{\mathbf{j}} |d_{\mathbf{j},\mathbf{s}}^*(f)|^p,$$

where

$$d_{\mathbf{j},\mathbf{s}}^*(f) := \int_{\mathbb{R}^d} f(x) \psi_{\mathbf{j},\mathbf{s}}^*(x) dx = 2^{-(r,\mathbf{s})} \int_{\mathbb{R}^d} D^r f(x) \cdot \mu_{\mathbf{j},\mathbf{s}}^* dx,$$

$\mu_{\mathbf{j},\mathbf{s}}^*$  is defined as  $\mu_{\mathbf{j},\mathbf{s}}$  in a same way. Using the same methods as Lemma 2.1, we can get

$$\|D_{\mathbf{s}}^* f\|_p^p \ll 2^{p(1/2-1/p)|\mathbf{s}|} \sum_{\mathbf{j}} |d_{\mathbf{j},\mathbf{s}}^*(f)|^p \ll 2^{-(r,\mathbf{s})p} \|D^r f\|_p^p. \quad \square$$



**Lemma 2.4.** For any  $f \in L_p(\mathbb{R}^d)$ ,  $1 \leq p \leq \infty$ ,  $\mathbf{s} \geq 0$ ,  $e := e(s) := \{j \in e_d: s_j > 0\}$ , then

$$\|D_{\mathbf{s}}^* f\|_p \ll \Omega^{l^e}(f, 2^{-\mathbf{s}^e})_p, \quad (2.6)$$

where  $2^{-\mathbf{s}} = (2^{-s_1}, \dots, 2^{-s_d})$ .

**Proof.** Without loss of generality, we suppose that  $\mathbf{s} > 0$ . For any  $G(x) \in L_p(\mathbb{R}^d)$ , we define

$$G_{l,j}^m(x) := l^l \int_0^{1/l} \cdots \int_0^{1/l} \sum_{i=1}^l (-1)^{i+l} \binom{l}{i} \\ \times G(x_1, \dots, x_j + i2^{-m}(u_1 + \cdots + u_l), \dots, x_d) du_1 \cdots du_l.$$

Then we have

$$G_{l,j}^m(x) - G(x) = l^l \int_0^{1/l} \cdots \int_0^{1/l} \Delta_{2^{-m}(u_1 + \cdots + u_l), j}^l G(x) du_1 \cdots du_l$$

and

$$\frac{\partial^l}{\partial x_j^l} G_{l,j}^m(x) = l^l 2^{ml} \sum_{i=1}^l (-1)^{i+l} \binom{l}{i} i^{-l} \Delta_{(i/l)2^{-m}, j}^l G(x).$$

Since

$$f(x) = (f(x) - f_{l,1}^{s_1}(x)) + f_{l,1}^{s_1}(x) := f_1^1(x) + f_1^2(x),$$

where  $f_1^1(x) := f(x) - f_{l,1}^{s_1}(x)$ ,  $f_1^2(x) := f_{l,1}^{s_2}(x)$ . We can also decompose  $f_1^1(x)$ ,  $f_1^2(x)$  as follows:

$$f_2^1(x) = f_1^1(x) - (f_1^1)_{l,2}^{s_2}(x), \quad f_2^2(x) = (f_1^1)_{l,2}^{s_2}(x), \\ f_2^3(x) = f_1^2(x) - (f_1^2)_{l,2}^{s_2}(x), \quad f_2^4(x) = (f_1^2)_{l,2}^{s_2}(x).$$

Continuing this process, we get sequences of functions  $f_d^1(x), f_d^2(x), \dots, f_d^{2^d}(x)$ , such that

$$f(x) = f_d^1(x) + f_d^2(x) + \cdots + f_d^{2^d}(x)$$

and for every  $j = 1, 2, \dots, 2^d$ , there exists a  $\beta^j = (\beta_1^j, \dots, \beta_d^j)$ ,  $\beta_i^j = 0$  or  $1$ , such that

$$\|D^{l\beta^j} f_d^j(\cdot)\|_p \ll 2^{l(s, \beta^j)} \Omega^l(f, 2^{-\mathbf{s}})_p.$$

By Lemma 2.3, we get

$$\|D_{\mathbf{s}}^* f_d^j(\cdot)\|_p \ll 2^{-l(s, \beta^j)} \|D^{l\beta^j} f_d^j(\cdot)\|_p \ll \Omega^l(f, 2^{-\mathbf{s}})_p.$$

Hence

$$\|D_{\mathbf{s}}^* f(\cdot)\|_p \leq \sum_{j=1}^{2^d} \|D_{\mathbf{s}}^* f_d^j(\cdot)\|_p \ll \Omega^l(f, 2^{-\mathbf{s}})_p. \quad \square$$

### 3. Representation theorems of the spaces $S_{p,\theta}^r B(\mathbb{R}^d)$ and $S_p^r L(\mathbb{R}^d)$ by hyperbolic wavelets

First we consider the space  $S_{p,\theta}^r B(\mathbb{R}^d)$ . For every  $f \in L_p(\mathbb{R}^d)$ ,  $1 \leq p \leq \infty$ , we have the decomposition

$$f(x) = \sum_{\mathbf{s} \geq 0} D_{\mathbf{s}}^* f(x), \quad D_{\mathbf{s}}^* f(x) = \sum_{\mathbf{j} \in \mathbb{Z}^d} d_{\mathbf{j},\mathbf{s}}^*(f) \psi_{\mathbf{j},\mathbf{s}}^*(x). \quad (3.1)$$

We use the formulas (3.1) to renorm the space  $S_{p,\theta}^r B(\mathbb{R}^d)$ . The next theorem (called the representation theorem by hyperbolic wavelets) is fundamental.

**Theorem 3.1.** *Let  $1 \leq p \leq \infty$ ,  $r = (r_1, \dots, r_d)$ ,  $0 < r_1, \dots, r_d < l$ . Then  $f \in S_{p,\theta}^r B(\mathbb{R}^d)$ , if and only if*

$$\left( \sum_{\mathbf{s} \geq 0} 2^{(r,\mathbf{s})\theta} \|D_{\mathbf{s}}^* f\|_p^\theta \right)^{1/\theta} \leq \infty, \quad 1 \leq \theta \leq \infty.$$

In this case, we have

$$\|f\|_{S_{p,\theta}^r B} \asymp \left( \sum_{\mathbf{s} \geq 0} 2^{(r,\mathbf{s})\theta} \|D_{\mathbf{s}}^* f\|_p^\theta \right)^{1/\theta}, \quad 1 \leq \theta < \infty. \quad (3.2)$$

When  $\theta = \infty$ , we have

$$\|f\|_{S_p^r H} = \|f\|_{S_{p,\infty}^r B} \asymp \sup_{\mathbf{s} \geq 0} 2^{(r,\mathbf{s})} \|D_{\mathbf{s}}^* f\|_p. \quad (3.3)$$

**Proof.** We only give the proof when  $1 \leq \theta < \infty$ . The proof of the case  $\theta = \infty$  is similar.

*Necessity.* For any  $\mathbf{s} > 0$ ,  $t = (t_1, \dots, t_d) > 0$ , we define

$$\beta := (\beta_1, \dots, \beta_d), \quad \beta_i := \begin{cases} 0, & \text{if } 2^{s_i} t_i \geq 1, \\ 1, & \text{otherwise,} \end{cases} \quad i = 1, \dots, d.$$

Let  $e_1 = \{j \in e_d: \beta_j = 0\}$ ,  $e_2 = \{j \in e_d: \beta_j = 1\}$ . Then from (2.4), we get

$$\begin{aligned} \sup_{|h_j| \leq t_j} \|\Delta_h^l(D_{\mathbf{s}}^* f)(\cdot)\|_p &\leq \sup_{|h_j| \leq t_j} 2^{l|e_1|} \left( \prod_{j \in e_2} |h_j|^l \right) \|D^{l\beta}(D_{\mathbf{s}}^* f)(\cdot)\|_p \\ &\ll \left( \prod_{j \in e_2} |t_j|^l \right) 2^{(l\beta,\mathbf{s})} \|D_{\mathbf{s}}^* f(\cdot)\|_p \\ &= \prod_{j=1}^d \min\{1, |t_j|^l 2^{s_j l}\} \|D_{\mathbf{s}}^* f(\cdot)\|_p. \end{aligned}$$

Since

$$\|f\|_{S_{p,\theta}^r B} = \sum_{e \subset e_d} \left( \int_0^2 \cdots \int_0^2 \prod_{j \in e} t_j^{-\theta r_j - 1} (\Omega^{l^e}(f, t^e)_p)^\theta \prod_{j \in e} dt_j \right)^{1/\theta},$$

without loss of generality, we may consider the term  $e = e_d$  only. Then we have

$$\begin{aligned} & \left( \int_0^2 \cdots \int_0^2 \prod_{j \in e} t_j^{-\theta r_j - 1} (\Omega^{l^e}(f, t^e)_p)^\theta \prod_{j \in e} dt_j \right)^{1/\theta} \\ &= \left( \sum_{\mathbf{k} \geq 0} \int_{2^{-k_1}}^{2^{-k_1+1}} \cdots \int_{2^{-k_d}}^{2^{-k_d+1}} \prod_{j=1}^d t_j^{-\theta r_j - 1} (\Omega^l(f, t)_p)^\theta \prod_{j=1}^d dt_j \right)^{1/\theta} \\ &\ll \left( \sum_{\mathbf{k} \geq 0} (\Omega^l(f, 2^{-\mathbf{k}})_p)^\theta 2^{\theta(r, \mathbf{k})} \right)^{1/\theta}. \end{aligned}$$

Then

$$\begin{aligned} (\Omega^l(f, 2^{-\mathbf{k}})_p)^\theta &\leq \left( \sup_{|h_j| \leq 2^{-k_j}} \|\Delta_h^l f(\cdot)\|_p \right)^\theta \leq \left( \sum_{\mathbf{s} \geq 0} \sup_{|h_j| \leq 2^{-k_j}} \|\Delta_h^l (D_{\mathbf{s}}^* f)(\cdot)\|_p \right)^\theta \\ &\ll \left( \sum_{\mathbf{s} \geq 0} \|D_{\mathbf{s}}^* f(\cdot)\|_p \prod_{j=1}^d \min\{1, 2^{(s_j - k_j)l}\} \right)^\theta \\ &= \left( \sum_{e \in e_d} \sum_{\mathbf{s} \in G(e, \mathbf{k})} \|D_{\mathbf{s}}^* f(\cdot)\|_p \prod_{j=1}^d \min\{1, 2^{(s_j - k_j)l}\} \right)^\theta, \end{aligned}$$

where  $G(e, \mathbf{k}) := \{\mathbf{s} \geq 0: s_j \leq k_j, \text{ if } j \in e; \text{ and } s_j \geq k_j, \text{ if } j \notin e\}$ . Choose  $\alpha_j, \beta_j$  such that  $0 < \alpha_j < r_j < \beta_j < l, j = 1, \dots, d$ . Below we suppose  $e_0 = \{1, \dots, m\}, 1 \leq m \leq d$ . It suffices to consider the term  $e_0$  only. In this case, we have

$$\begin{aligned} & \left( \sum_{\mathbf{s} \in G(e_0, \mathbf{k})} \|D_{\mathbf{s}}^* f\|_p \prod_{j=1}^d \min\{1, 2^{(s_j - k_j)l}\} \right)^\theta \\ &= \left( \sum_{\substack{s_j \leq k_j \\ j=1, \dots, m}} \prod_{j=1}^m 2^{(s_j - k_j)l} \sum_{\substack{s_j > k_j \\ j=m+1, \dots, d}} \|D_{\mathbf{s}}^* f\|_p \right)^\theta \\ &= \left( \sum_{\substack{s_j \leq k_j \\ j=1, \dots, m}} \prod_{j=1}^m 2^{\beta_j s_j - k_j l} \cdot 2^{(l - \beta_j)s_j} \sum_{\substack{s_j > k_j \\ j=m+1, \dots, d}} \|D_{\mathbf{s}}^* f\|_p \prod_{j=m+1}^d 2^{\alpha_j s_j} \cdot 2^{-\alpha_j s_j} \right)^\theta \\ &\leq \left( \sum_{\substack{s_j \leq k_j \\ j=1, \dots, m}} \prod_{j=1}^m 2^{(\beta_j s_j - k_j l)\theta} \sum_{\substack{s_j > k_j \\ j=m+1, \dots, d}} \|D_{\mathbf{s}}^* f\|_p^\theta \prod_{j=m+1}^d 2^{\alpha_j s_j \theta} \right) \\ &\quad \cdot \left( \sum_{\substack{s_j \leq k_j \\ j=1, \dots, m}} \prod_{j=1}^m 2^{(l - \beta_j)s_j \theta'} \sum_{\substack{s_j > k_j \\ j=m+1, \dots, d}} \prod_{j=m+1}^d 2^{-\alpha_j s_j \theta'} \right)^{\theta/\theta'} \end{aligned}$$

$$\ll \sum_{\substack{s_j \leq k_j \\ j=1, \dots, m}} \prod_{j=1}^m 2^{(s_j - k_j)\beta_j \theta} \sum_{\substack{s_j > k_j \\ j=m+1, \dots, d}} \|D_{\mathbf{s}}^* f\|_p^\theta \prod_{j=m+1}^d 2^{(s_j - k_j)\alpha_j \theta}.$$

Hence

$$\begin{aligned} & \sum_{\mathbf{k} \geq 0} 2^{(\mathbf{k}, r)\theta} \sum_{\substack{s_j \leq k_j \\ j=1, \dots, m}} \prod_{j=1}^m 2^{(s_j - k_j)\beta_j \theta} \sum_{\substack{s_j > k_j \\ j=m+1, \dots, d}} \|D_{\mathbf{s}}^* f\|_p^\theta \prod_{j=m+1}^d 2^{(s_j - k_j)\alpha_j \theta} \\ &= \sum_{\mathbf{s} \geq 0} \|D_{\mathbf{s}}^* f\|_p^\theta \sum_{\substack{k_j \geq s_j \\ j=1, \dots, m}} 2^{(\mathbf{k}, r)\theta} \prod_{j=1}^m 2^{(s_j - k_j)\beta_j \theta} \sum_{\substack{k_j < s_j \\ j=m+1, \dots, d}} \prod_{j=m+1}^d 2^{(s_j - k_j)\alpha_j \theta} \\ &\ll \sum_{\mathbf{s} \geq 0} 2^{(r, \mathbf{s})\theta} \|D_{\mathbf{s}}^* f\|_p^\theta. \end{aligned}$$

*Sufficiency.* From Lemma 2.4, we know for any  $\mathbf{s} \geq 0$ , we have

$$\|D_{\mathbf{s}}^* f\|_p \ll \Omega^{l^e}(f, (2^{-\mathbf{s}})^e)_p,$$

where  $e := e(\mathbf{s}) := \{j \in e_d : s_j > 0\}$ . Hence

$$\begin{aligned} \left( \sum_{\mathbf{s} \geq 0} 2^{(\mathbf{s}, r)\theta} \|D_{\mathbf{s}}^* f\|_p^\theta \right)^{1/\theta} &= \left( \sum_{e \subset e_d} \sum_{\mathbf{s}^e > 0} 2^{(\mathbf{s}^e, r)\theta} \|D_{\mathbf{s}^e}^* f\|_p^\theta \right)^{1/\theta} \\ &\ll \left( \sum_{e \subset e_d} \sum_{\mathbf{s}^e > 0} 2^{(\mathbf{s}^e, r)\theta} \Omega^{l^e}(f, (2^{-\mathbf{s}})^e)_p^\theta \right)^{1/\theta} \\ &\ll \left( \sum_{e \subset e_d} \int_0^2 \cdots \int_0^2 \prod_{j \in e} t_j^{-\theta r_j - 1} (\Omega^{l^e}(f, t^e)_p)^\theta \prod_{j \in e} dt_j \right)^{1/\theta} \\ &\ll \|f\|_{S_{p, \theta}^r B}. \quad \square \end{aligned}$$

**Remark 3.1.** We can use the decomposition

$$f(x) = \sum_{\mathbf{s} \geq 0} \sum_{\mathbf{j} \in \mathbb{Z}^d} d_{\mathbf{j}, \mathbf{s}}^*(f) \psi_{\mathbf{j}, \mathbf{s}}^*(x), \quad d_{\mathbf{j}, \mathbf{s}}^*(f) = \int_{\mathbb{R}^d} f(x) \psi_{\mathbf{j}, \mathbf{s}}^*(x) dx \quad (3.4)$$

to give a characterization of the space  $S_{p, \theta}^r B(\mathbb{R}^d)$ . From the above theorem and Lemma 2.1, we obtain the following theorem.

**Theorem 3.1'.** Let  $1 \leq p \leq \infty$ ,  $r = (r_1, \dots, r_d)$ ,  $0 < r_1, \dots, r_d < l$ . Then for any  $f \in S_{p, \theta}^r B(\mathbb{R}^d)$ ,  $1 \leq \theta < \infty$ , we have

$$\|f\|_{S_{p, \theta}^r B} \asymp \left( \sum_{\mathbf{s} \geq 0} 2^{((r, \mathbf{s}) + (1/2 - 1/p)|\mathbf{s}|)\theta} \left( \sum_{\mathbf{j} \in \mathbb{Z}^d} |d_{\mathbf{j}, \mathbf{s}}^*(f)|^p \right)^{\theta/p} \right)^{1/\theta}. \quad (3.5)$$

When  $\theta = \infty$ , we have

$$\|f\|_{S_p^r H} \asymp \sup_{s \geq 0} 2^{(r, s) + (1/2 - 1/p)|s|} \left( \sum_{j \in \mathbb{Z}^d} |d_{j, s}^*(f)|^p \right)^{1/p}. \quad (3.6)$$

**Remark 3.2.** We cannot use the decomposition

$$f(x) = \sum_{\mathbf{I} \in \mathcal{D}(\mathbb{R}^d)} d_{\mathbf{I}}(f) \psi_{\mathbf{I}}(x), \quad d_{\mathbf{I}}(f) = \int_{\mathbb{R}^d} f(x) \psi_{\mathbf{I}}(x) dx, \quad (3.7)$$

or

$$f(x) = \sum_{\mathbf{k} \in \mathbb{Z}^d} D_{\mathbf{k}} f(x), \quad D_{\mathbf{k}} f(x) = \sum_{j \in \mathbb{Z}^d} d_{j, \mathbf{k}}(f) \psi_{j, \mathbf{k}}(x) \quad (3.8)$$

to give an equivalent description of the norm  $\|f\|_{S_{p, \theta}^r B}$ , since the space  $S_{p, \theta}^r B(\mathbb{R}^d)$  is not a homogeneous space.

We now turn to the decompositional description of spaces  $S_p^r L(\mathbb{R}^d)$ ,  $1 < p < \infty$ . First, we introduce some notations. Let  $\mathcal{D} \subset \mathcal{D}(\mathbb{R})$  be an index set. Then for univariate functions  $f^1, \dots, f^d$ , and  $\mathbf{I} = I_1 \times \dots \times I_d \in \mathcal{D}^d$ , we use the notation  $f_{\mathbf{I}}(x) := \prod_{i=1}^d f_{I_i}^i(x_i)$  the  $L_2(\mathbb{R}^d)$  normalized shifted dilate of functions  $f^1, \dots, f^d$  (or  $f$  if  $f^1 = \dots = f^d = f$ ). For  $1 < p < \infty$ , we say that a family of real valued functions  $f_{\mathbf{I}}(x)$ ,  $\mathbf{I} \in \mathcal{D}^d$ , satisfies the Littlewood–Paley property (we briefly write *LPP*) for  $p$ , if for any finite sequence  $(c_{\mathbf{I}})$  of real number, we have

$$\left\| \sum_{\mathbf{I} \in \mathcal{D}^d} c_{\mathbf{I}} f_{\mathbf{I}}(\cdot) \right\|_p \asymp \left\| \left( \sum_{\mathbf{I} \in \mathcal{D}^d} |c_{\mathbf{I}} f_{\mathbf{I}}(\cdot)|^2 \right)^{1/2} \right\|_p. \quad (3.9)$$

We also say  $f_{\mathbf{I}}(x)$ ,  $\mathbf{I} \in \mathcal{D}^d$  satisfies the strong Littlewood–Paley property (we briefly write *SLPP*) for  $p$ , if for any finite sequence  $(c_{\mathbf{I}})$  of real number, we have

$$\left\| \sum_{\mathbf{I} \in \mathcal{D}^d} c_{\mathbf{I}} f_{\mathbf{I}}(\cdot) \right\|_p \asymp \left\| \left( \sum_{\mathbf{I} \in \mathcal{D}^d} |c_{\mathbf{I}} \chi_{\mathbf{I}}|^2 \right)^{1/2} \right\|_p, \quad (3.10)$$

where  $\chi$  is the characteristic function of  $[0, 1]$ , and  $\chi_{\mathbf{I}}$  is the  $L_2(\mathbb{R}^d)$  normalized shifted dilates of function  $\chi$ . From [4], we know that if

$$f^i(x) \leq c(1 + |x|)^{-1}, \quad \text{a.e. } x \in \mathbb{R} \text{ and } f^i \text{ is a nonzero function, } i = 1, \dots, d, \quad (3.11)$$

then  $f_{\mathbf{I}}$ ,  $\mathbf{I} \in \mathcal{D}$  satisfies *LPP* if and only if  $f_{\mathbf{I}}$ ,  $\mathbf{I} \in \mathcal{D}$  satisfies *SLPP*.

If for any  $f \in L_p(\mathbb{R}^d)$ , we have a unique representation  $f(x) = \sum_{\mathbf{I} \in \mathcal{D}^d} c_{\mathbf{I}} f_{\mathbf{I}}(x)$ , and for any assignment  $\varepsilon_{\mathbf{I}} = \pm 1$ ,  $\mathbf{I} \in \mathcal{D}^d$ , we have

$$\left\| \sum_{\mathbf{I} \in \mathcal{D}^d} c_{\mathbf{I}} f_{\mathbf{I}}(\cdot) \right\|_p \asymp \left\| \sum_{\mathbf{I} \in \mathcal{D}^d} \varepsilon_{\mathbf{I}} c_{\mathbf{I}} f_{\mathbf{I}}(\cdot) \right\|_p, \quad (3.12)$$

then we say  $f_{\mathbf{I}}, \mathbf{I} \in \mathcal{D}^d$  is a unconditional basis of  $L_p(\mathbb{R}^d)$ . Applying Khinchine's inequality, we can get that (3.9) and (3.12) are equivalent (see [4]). Using the same methods in the proof of [4, Lemma 2.1, p. 7–8], we obtain the following lemma.

**Lemma 3.1.** *If  $\{f_I^i\}_{I \in \mathcal{D}}$  ( $i = 1, \dots, d$ ) satisfy LPP in  $L_p(\mathbb{R})$  for some  $p$ ,  $1 < p < \infty$ , then the multivariate family  $f_{\mathbf{I}}, \mathbf{I} \in \mathcal{D}^d$  also satisfies LPP for this  $p$ .*

Since  $\psi$  is an  $l$ -regular wavelet, we take for granted the known facts that for  $1 < p < \infty$  and any  $r = (r_1, \dots, r_d)$ ,  $0 \leq r_i < l$ , the families of functions  $(\psi^{(r_i)})_I, I \in \mathcal{D}(\mathbb{R})$  and  $((\psi^*)^{(r_i)})_I, I \in \mathcal{D}_+(\mathbb{R})$  ( $i = 1, \dots, d$ ) satisfy LPP in  $L_p(\mathbb{R})$  (see [9]). Hence by Lemma 3.1, we know that  $(D^r \psi)_{\mathbf{I}}, \mathbf{I} \in \mathcal{D}(R^d)$  and  $(D^r \psi^*)_{\mathbf{I}}, \mathbf{I} \in \mathcal{D}_+(R^d)$  satisfy LPP. It is obvious that  $\psi^{(r_i)}$  and  $(\psi^*)^{(r_i)}$  satisfy (3.11), so  $(D^r \psi)_{\mathbf{I}}, \mathbf{I} \in \mathcal{D}(R^d)$  and  $(D^r \psi^*)_{\mathbf{I}}, \mathbf{I} \in \mathcal{D}_+(R^d)$  satisfy SLPP. From this and the definition of the norm of  $\|f\|_{S_p^r L}$ , using the decomposition (3.7), we get the following results.

**Theorem 3.2.** *Let  $\psi$  be an  $l$ -regular univariate wavelet,  $r = (r_1, \dots, r_d)$ ,  $0 < r_i < l$ . Then for any  $f \in S_p^r L(\mathbb{R}^d)$ , we have*

$$\|f\|_{S_p^r L} \asymp \left\| \left( \sum_{\mathbf{I} \in \mathcal{D}(\mathbb{R}^d)} \prod_{j=1}^d (1 + |I_j|^{-2r_j}) \cdot (|d_{\mathbf{I}}(f)| \chi_{\mathbf{I}}(\cdot))^2 \right)^{1/2} \right\|_p. \quad (3.13)$$

**Remark 3.3.** Similarly, we can use (3.4) to express the equivalent norms of  $\|f\|_{S_p^r L}$ . And we get the following theorem.

**Theorem 3.2'.** *Let  $\psi$  be an  $l$ -regular univariate wavelet,  $r = (r_1, \dots, r_d)$ ,  $0 < r_i < l$ . Then for any  $f \in S_p^r L$ , we have*

$$\|f\|_{S_p^r L} \asymp \left\| \left( \sum_{\mathbf{s} \geq 0} \sum_{\mathbf{j} \in \mathbb{Z}^d} (2^{(r, \mathbf{s})} |d_{\mathbf{j}, \mathbf{s}}^*(f)| \chi_{\mathbf{j}, \mathbf{s}}(\cdot))^2 \right)^{1/2} \right\|_p. \quad (3.14)$$

In the following, we use the decomposition  $f(x) = \sum_{\mathbf{s} \geq 0} D_{\mathbf{s}}^* f(x)$ . Since the family of functions  $\psi_{\mathbf{j}, \mathbf{s}}^*, \mathbf{j}, \mathbf{s} \in \mathbb{Z}^d, \mathbf{s} \geq 0$  satisfies LPP for  $p$ ,  $1 < p < \infty$ , then for any assignment  $\varepsilon_{\mathbf{s}} = \pm 1$ , we have

$$\left\| \sum_{\mathbf{s} \geq 0} \varepsilon_{\mathbf{s}} D_{\mathbf{s}}^* f(\cdot) \right\|_p \asymp \left\| \sum_{\mathbf{s} \geq 0} D_{\mathbf{s}}^* f(\cdot) \right\|_p \asymp \left\| \left( \sum_{\mathbf{s} \geq 0} \sum_{\mathbf{j} \in \mathbb{Z}^d} |d_{\mathbf{j}, \mathbf{s}}^*(f) \psi_{\mathbf{j}, \mathbf{s}}^*(\cdot)|^2 \right)^{1/2} \right\|_p. \quad (3.15)$$

From (3.15) and Khinchine's inequality, we can get the following result:

$$\|f(\cdot)\|_p = \left\| \sum_{\mathbf{s} \geq 0} D_{\mathbf{s}}^* f(\cdot) \right\|_p \asymp \left\| \left( \sum_{\mathbf{s} \geq 0} |D_{\mathbf{s}}^* f(\cdot)|^2 \right)^{1/2} \right\|_p. \quad (3.16)$$

Hence, by (3.14) and (3.16), we obtain the following theorem.

**Theorem 3.3.** Let  $\psi$  be an  $l$ -regular univariate wavelet,  $r = (r_1, \dots, r_d)$ ,  $0 < r_i < l$ . Then for any  $f \in S_p^r L(\mathbb{R}^d)$ ,  $1 < p < \infty$ , we have

$$\|f\|_{S_p^r L} \asymp \left\| \left( \sum_{s \geq 0} |2^{(r,s)} D_s^* f(\cdot)|^2 \right)^{1/2} \right\|_p. \quad (3.17)$$

**Remark 3.4.** We may use the decomposition (3.8) to give an equivalent norm of  $\|f\|_{S_p^r L}$ . Using (3.13) and the above induction, we get the following theorem.

**Theorem 3.3'.** Let  $\psi$  be  $l$ -regular univariate wavelet,  $r = (r_1, \dots, r_d)$ ,  $0 < r_i < l$ . Then for any  $f \in S_p^r L(\mathbb{R}^d)$ ,  $1 < p < \infty$ , we have

$$\|f\|_{S_p^r L} \asymp \left\| \left( \sum_{\mathbf{k} \in \mathbb{R}^d} \prod_{i=1}^d (1 + 4^{r_i k_i}) \cdot |D_{\mathbf{k}} f(\cdot)|^2 \right)^{1/2} \right\|_p. \quad (3.18)$$

#### 4. Approximation by hyperbolic wavelets

In this section, we always suppose that  $\varphi, \psi$  are  $l$ -regular,  $r = (r_1, \dots, r_d)$ ,  $r = r_1 \gamma$ ,  $0 < r_1, \dots, r_d < l$ ,  $\gamma$  and  $\gamma'$  satisfy  $1 = \gamma_1 = \gamma'_1 = \dots = \gamma_v = \gamma'_v$  and  $1 < \gamma'_j < \gamma_j$ ,  $j = v + 1, \dots, d$  ( $1 \leq v \leq d$ ), for the purpose of simplification.

**Theorem 4.1.** Let  $1 < p \leq q < \infty$ ,  $1 \leq \theta \leq \infty$ . Then

$$E_n^\gamma(S_{q,\theta}^r B)_q \asymp E_n^{\gamma,*}(S_{q,\theta}^r B)_q \asymp \begin{cases} 2^{-nr_1 n^{(d-1)(1/q-1/\theta)_+}}, & 1 < q \leq 2, \\ 2^{-nr_1 n^{(d-1)(1/2-1/\theta)_+}}, & 2 < q \leq \infty, \end{cases} \quad (4.1)$$

$$E_n^\gamma(S_{p,\theta}^r B)_q \asymp E_n^{\gamma,*}(S_{p,\theta}^r B)_q \asymp 2^{-n(r_1-1/p+1/q)} n^{(v-1)(1/q-1/\theta)_+}, \quad p < q, \quad (4.2)$$

$$E_n^\gamma(S_p^r L)_q \asymp E_n^{\gamma,*}(S_p^r L)_q \asymp 2^{-n(r_1-1/p+1/q)}, \quad (4.3)$$

where  $a_+ = a$ ,  $a > 0$ ,  $a_+ = 0$ ,  $a \leq 0$ .

**Proof.** We can proceed in the same line as in [15] or in [13,14] to obtain the upper estimates of  $E_n^{\gamma,*}(S_{q,\theta}^r B)_q$ ,  $E_n^{\gamma,*}(S_{p,\theta}^r B)_q$ ,  $E_n^{\gamma,*}(S_p^r W)_q$ , we omit its details. By (1.16), we know it is enough to get the lower estimates of  $E_n^\gamma(S_{q,\theta}^r B)_q$ ,  $E_n^\gamma(S_{p,\theta}^r B)_q$ ,  $E_n^\gamma(S_p^r W)_q$ . First we construct two functions  $g_s(x)$  and  $h_s(x)$ . For any  $\mathbf{s} \geq 0$ ,  $\rho(\mathbf{s}) := \{\mathbf{k} \in \mathbb{Z}^d: 0 \leq k_j \leq 2^{s_j} - 1, j = 1, \dots, d\}$ ,  $\omega = (1, \dots, 1) \in \mathbb{R}^d$ , we define

$$g_s(x) := 2^{|s|/2} \psi_{\omega, \mathbf{s}}(x), \quad h_s(x) := 2^{-|s|/2} \sum_{\mathbf{j} \in \rho(\mathbf{s})} \psi_{\mathbf{j}, \mathbf{s}}(x).$$

It is easy to know that  $g_s, h_s \in W_s(p) \subset W_s^*(p)$  and that for  $1 \leq p \leq \infty$ ,  $1/p + 1/p' = 1$ , we have

$$\|g_s\|_p \asymp 2^{|s|/p'}, \quad \|h_s\|_p \asymp 1. \quad \square \quad (4.4)$$

**Lemma 4.1.** Let  $1 < p < \infty$  and  $f(x) = \sum_{s \geq 0} c_s h_s(x)$ . Then

$$\|f\|_p \asymp \left( \sum_{s \geq 0} |c_s|^2 \right)^{1/2}. \quad (4.5)$$

**Proof.** As  $\psi_I, I \in \mathcal{D}(\mathbb{R}^d)$  satisfies *SLPP*, we have

$$\|f(\cdot)\|_p \asymp \left\| \left( \sum_{s \geq 0} \sum_{\mathbf{k} \in \rho(\mathbf{s})} |c_s|^2 2^{-|s|} |\chi_{\mathbf{k}, \mathbf{s}}(\cdot)|^2 \right)^{1/2} \right\|_p.$$

Since

$$2^{-|s|} \sum_{\mathbf{k} \in \rho(\mathbf{s})} |\chi_{\mathbf{k}, \mathbf{s}}(x)|^2 = \begin{cases} 1, & x \in [0, 1]^d, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we obtain

$$\|f\|_p \asymp \left( \sum_{s \geq 0} |c_s|^2 \right)^{1/2}. \quad \square$$

**Lemma 4.2.** Suppose that  $1 < p < \infty$ , and that  $f(x) = \sum_{s \geq 0} d_s g_s(x)$ . Then

$$\|f\|_p \asymp \left( \sum_{s \geq 0} |d_s|^p 2^{(p/p')|s|} \right)^{1/p}. \quad (4.6)$$

**Proof.** As  $\psi_I, I \in \mathcal{D}(\mathbb{R}^d)$  satisfies *SLPP*, we get

$$\|f(\cdot)\|_p \asymp \left\| \left( \sum_{s \geq 0} |d_s|^2 2^{|s|} |\chi_{\omega, \mathbf{s}}(\cdot)|^2 \right)^{1/2} \right\|_p.$$

Since the support set of the functions  $\chi_{\omega, \mathbf{s}}$  is disjoint, so we obtain

$$\begin{aligned} \|f(\cdot)\|_p^p &\asymp \int_{\mathbb{R}^d} \left( \sum_{s \geq 0} |d_s|^2 2^{|s|} |\chi_{\omega, \mathbf{s}}(x)|^2 \right)^{p/2} dx \\ &= \int_{\mathbb{R}^d} \left( \sum_{s \geq 0} |d_s|^p 2^{p|s|/2} |\chi_{\omega, \mathbf{s}}(x)|^p \right) dx \asymp \sum_{s \geq 0} |d_s|^p 2^{p/p'|s|}. \end{aligned}$$

We begin our discussion above the lower estimates. First we give the proof of (4.1) for  $1 < q \leq 2, q \leq \theta \leq \infty$ . Consider the function

$$f_{rq\theta}(x) = n^{-(d-1)/\theta} \cdot 2^{r_1 n} \sum_{(\gamma, s) > n} 2^{-2(r, s) - |s|/q'} g_s(x). \quad (4.7)$$

By Theorem 3.1 and (4.4), we know



$$\begin{aligned}\|f_{rq\theta}\|_{S_{q,\theta}^r B} &\asymp n^{-(d-1)/\theta} \cdot 2^{r_1 n} \left( \sum_{(\gamma,s)>n} 2^{(r,s)\theta} \cdot 2^{(-2(r,s)-|s|/q')\theta} \cdot \|g_s\|_q^\theta \right)^{1/\theta} \\ &\asymp 2^{nr_1} \cdot n^{-(d-1)/\theta} \cdot \left( \sum_{(\gamma,s)>n} 2^{-(r,s)\theta} \right)^{1/\theta} \asymp 1.\end{aligned}$$

Consequently, by (4.6), we get

$$\begin{aligned}E_n^\gamma(S_{q,\theta}^r B)_q &\gg E_n^\gamma(f_{rq\theta})_q \asymp \|f_{rq\theta} - S_n^\gamma f_{rq\theta}\|_q = \|f_{rq\theta}\|_q \\ &\asymp n^{-(d-1)/\theta} \cdot 2^{r_1 n} \cdot \left( \sum_{(\gamma,s)>n} 2^{(-2(r,s)-|s|/q')q} \cdot 2^{(|s|/q')q} \right)^{1/q} \\ &= n^{-(d-1)/\theta} \cdot 2^{r_1 n} \cdot \left( \sum_{(\gamma,s)>n} 2^{-2q(r,s)} \right)^{1/q} \asymp 2^{-r_1 n} \cdot n^{(d-1)(1/q-1/\theta)}.\end{aligned}$$

For the other case, the proof is similar. We only give the extreme functions which attain the corresponding approximating order.

*Case 2.* Let  $2 < q < \infty$ ,  $2 \leq \theta \leq \infty$ . The extreme function of  $E_n^\gamma(S_{q,\theta}^r B)_q$  is

$$\psi_{r,\theta}(x) = 2^{r_1 n} \cdot n^{-(d-1)/\theta} \sum_{(s,\gamma)>n} 2^{-2(r,s)} h_s(x). \quad (4.8)$$

*Case 3.* Let  $1 < q \leq 2$ ,  $1 \leq \theta < q$ , or  $2 < q < \infty$ ,  $1 \leq \theta \leq 2$ . The extreme function of  $E_n^\gamma(S_{q,\theta}^r B)_q$  is

$$\tilde{\psi}_r(x) = 2^{-r_1 n} h_{\tilde{s}}(x), \quad (4.9)$$

where  $\tilde{s} = (n+1, 0, \dots, 0)$ .

*Case 4.* Let  $1 < p < q < \infty$ ,  $1 \leq \theta \leq q$ . The extreme function of  $E_n^\gamma(S_{p,\theta}^r B)_q$  is

$$\tilde{g}_{pr}(x) = 2^{-r_1 n - |\tilde{s}|/p'} g_{\tilde{s}}(x). \quad (4.10)$$

*Case 5.* Let  $1 < p < q < \infty$ ,  $q \leq \theta \leq \infty$ . The extreme function of  $E_n^\gamma(S_{p,\theta}^r B)_q$  is

$$g_{rp\theta}(x) = 2^{r_1 n} \cdot n^{-(v-1)/\theta} \sum_{(\gamma',s)>n} 2^{-2(r,s)-|s|/p'} g_s(x). \quad (4.11)$$

*Case 6.* Let  $1 < p \leq q < \infty$ . The extreme function of  $E_n^\gamma(S_p^r L)_q$  is

$$\tilde{g}_{pr}(x) = 2^{-r_1 n - |\tilde{s}|/p'} g_{\tilde{s}}(x). \quad (4.12)$$

The theorem is proved.  $\square$

Using the same methods, we can get the following theorem.

**Theorem 4.1'.** Let  $1 < p \leq q < \infty$ ,  $1 \leq \theta \leq \infty$ . Then

$$\begin{aligned}
E_n^{\gamma'}(S_{q,\theta}^r B)_q &\asymp E_n^{\gamma',*}(S_{q,\theta}^r B)_q \asymp \begin{cases} 2^{-nr_1} n^{(v-1)(1/q-1/\theta)_+}, & 1 < q \leq 2, \\ 2^{-nr_1} n^{(v-1)(1/2-1/\theta)_+}, & 2 < q \leq \infty, \end{cases} \\
E_n^{\gamma'}(S_{p,\theta}^r B)_q &\asymp E_n^{\gamma',*}(S_{p,\theta}^r B)_q \asymp 2^{-n(r_1-1/p+1/q)} n^{(v-1)(1/q-1/\theta)_+}, \quad p < q, \\
E_n^{\gamma'}(S_p^r L)_q &\asymp E_n^{\gamma',*}(S_p^r L)_q \asymp 2^{-n(r_1-1/p+1/q)}.
\end{aligned}$$

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